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The explicit form of the Lie algebra of Wahlquist and Estabrook. A presentation problem.

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ABSTRACT

The structure of the KdV-Lie algebra of Wahlquist and Estabrook is made explicit. This is done with help of a table of Lie-products and an inherent grading of the algebra.

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Research blossoms in a good climate. This climate was created for me by my colleagues and friends Dr. P. Gragert, Dr. P. Jonker, Prof. dr. R. Martini and Prof. dr. P.J. Zandbergen. Their vivid interest and critical remarks were highly stimulating. The typing was done by the very capable hands of Ms. Joke Heisterkamp.

§ 1. INTRODUCTION

Wahlquist and Estabrook constructed in 1975 in their famous article a Liealgebra to the Korteweg-de Vries equation (see ref. (1)). This algebra was generated by the nine letters $x_1,\dots,x_9$ subjected to the following relations:

$$(1)\qquad \begin{cases} [x_1,x_3]=[x_2,x_3]=[x_1,x_4]=[x_2,x_6]=0, \\ x_7=[x_2,x_1],\ x_5=[x_1,x_7],\ x_6=[x_2,x_7],\ x_8=[x_4,x_3],\ x_9=[x_4,x_2] \text{ and} \\ [x_1,x_5]=x_9 \text{ and } [x_1,x_6]=-x_7+x_8. \end{cases}$$

With the extra relation

$$(2)\qquad x_9=\lambda(x_7-x_8),$$

λ being a (complex) parameter, they found a eightdimensional algebra. This is listed in table 1.

Table 1

$[x_1,x_2]=-x_7$	$[x_2,x_5]=-x_7+x_8$	$[x_4,x_7]=-\lambda x_5$
$[x_1,x_5]=\lambda(x_7-x_8)$	$[x_2,x_7]=x_6$	$[x_5,x_6]=x_7-x_8$
$[x_1,x_6]=-x_7+x_8$	$[x_3,x_4]=-x_8$	$[x_5,x_7]=-x_5-\lambda x_6$
$[x_1,x_7]=x_5$	$[x_4,x_5]=-\lambda^2(x_7-x_8)$	$[x_6,x_7]=x_6$
$[x_2,x_4]=-\lambda(x_7-x_8)$	$[x_4,x_6]=\lambda(x_7-x_8)$	

the other commutators being zero.

By defining

(3) $\quad x_{10}=[x_5,x_7],\ x_{11}=[x_4,x_7]$ and $x_{14}=[x_4,x_5]$

and adding the extra relation

(4) $\quad x_{14}=\lambda(x_7-x_8)+\mu x_9$

in stead of relation (2) Gragert was able with help of a by himself developed computer program to find a eleventh-dimensional algebra and by defining

(5) $\quad x_{12}=[x_5,x_9],\ x_{13}=[x_4,x_9]$ and $x_{15}=[x_{12},x_{11}]$

and adding the relation

(6) $\quad x_{15}=\lambda(x_7-x_8)+\mu x_9+\nu x_{14}$

in stead of (2) and (4) he even found a fourteenth-dimensional algebra. (see ref. (2)).

Just recently Gragert made a seventeenth-dimensional algebra meeting a request of Estabrook. This algebra enabled Estabrook to unravel independently the structure of EW (private communication).

As Shadwick has proven that the algebra of the words formed by the letters $x_1,\ldots,x_9$ subjected to the relations (1) is infinite dimensional (ref. (3)), it is clear that resort must be taken to pencil and paper to make the algebra explicit, although the computer is a powerful aid. Following Manin (ref. (4)), we shall denote this algebra by EW. We close this introduction with a description of the contents of the following sections.

In section 2 we present the tools that are used to make EW. In section 3 we shall prove that

$$EW=H\times(x_7-x_8)$$

where H is a five-dimensional solvable ideal, isomorphic with the algebra H of Shadwick (ref. (3)) and (x_7-x_8) is the ideal generated by x_7-x_8. The product is direct.

In section 4 we shall prove that

$$(x_7)=A\times_D(x_9)$$

where A is a $\mathfrak{sl}2$. The product is semidirect.

In section 5 we shall give (x_9) an explicit form.

It will turn out that (x_9) has a somewhat other shape than generally expected.

§ 2. PRELIMINARIES

In the sequel we shall write

xy in stead of $[x,y]$

$x(yz)$ in stead of $[x,[y,z]]$ etc.

The expression

$$x^n y$$

is inductively defined by

$$x^0 y = y \text{ and } x^{n+1}y = x(x^n y).$$

The algebra EW is the quotient of the *free Liealgebra* $L(x_1,\dots,x_9)$ and the ideal generated by the *relators*

$$(7) \qquad \begin{cases} x_7 - x_2x_1,\ x_5 - x_1x_7,\ x_6 - x_2x_7,\ x_8 - x_4x_3,\ x_9 - x_4x_2, \\ x_1x_3,\ x_1x_4,\ x_2x_3,\ x_2x_6,\ x_1x_5 - x_9 \text{ and } x_1x_6 + x_7 - x_8. \end{cases}$$

In ref. (1) it is made clear that the first line of relators are only *defining* relators. In other words: the letters $x_5,\dots,x_9$ are introduced for reasons of convenience. On the other hand $x_1,\dots,x_4$ stem from the KdV equation and are basic.

If we omit the letters $x_5,\dots,x_9$ we are left with the relators

$$(8) \qquad x_1x_3,\ x_1x_4,\ x_2x_3,\ x_2^3x_1,\ x_1^3x_2 - x_2x_4 \text{ and } x_1(x_2^2x_1) - x_1x_2 + x_3x_4.$$

Let EW' be the quotient of the free Liealgebra $L(y_1,\dots,y_4)$ and the ideal generated by (8) where "x" is replaced by "y".

PROPOSITION 1. *EW and EW' are isomorphic.*

PROOF. There is one Liealgebramorphism $\phi : L(x_1,\dots,x_9) \to EW'$ sending x_1 to $y_1,\dots,x_4$ to y_4, x_7 to y_2y_1, x_5 to $-y_1^2y_2$, x_6 to $y_2^2y_1$, x_8 to y_4y_3 and x_9 to y_4y_2 (see ref. (5)).

ϕ is zero at the ideal generated by (7) so there is a morphism $\phi' : EW \to EW'$.

On the other hand there is one morphism $\psi : L(y_1,\dots,y_4) \to EW$ sending y_1 to $x_1,\dots,y_4$ to x_4 and ψ is zero at the ideal generated by (8) so there is a morphism $\psi' : EW' \to EW$. It is easily seen that ϕ' and ψ' are each other's inverse.

Proposition 1 says that the introduction of extra letters is allowed. By suitable definition of extra letters $x_{10},\dots,x_{17}$ and use of the relators (8) Gragert produced the following table of Lieproducts which hold in EW.

It will become obvious in the following sections that this table is the main tool to analyse the structure of EW.

The letters $x_1, \ldots, x_4$ have a close relationship with the KdV equation. It is therefore to expected that $x_1, \ldots, x_4$ have a *physical dimension*. From a more algebraic point of view it is to expected that EW has a "natural" *grading* with the degrees being elements of a commutative (additively written) group, such that the relators (8) are all homogeneous (see ref. (6)).

PROPOSITION 2. EW has a $\mathbf{Z}$-grading with

$$\deg(x_1)=1,\ \deg(x_2)=-1,\ \deg(x_3)=-3 \text{ and } \deg(x_4)=3.$$

PROOF. Let $\deg(x_i)=\delta_i$, $1 \le i \le 4$. The first four relators (8) are always homogeneous. To make the other two homogeneous there must hold:

$$3\delta_1+\delta_2=\delta_2+\delta_4$$

$$2\delta_1+2\delta_2=\delta_1+\delta_2=\delta_3+\delta_4$$

yielding $\delta_2=-\delta_1$, $\delta_3=-3\delta_1$ and $\delta_4=3\delta_1$. One can then take $\delta_1=1$.

REMARK. This grading gives rise to an automorphism $\hat{\phi}_s$ defined by

$$\hat{\phi}_s(x_1)=\lambda x_1 \qquad \hat{\phi}_s(x_2)=\lambda^{-1}x_2$$

$$\hat{\phi}_s(x_3)=\lambda^{-3}x_3 \qquad \hat{\phi}_s(x_4)=\lambda^3 x_4$$

as already shown by Shadwick (ref. (3)).

The letter "s" means "scaling", λ is a complex parameter.

If EW^n is the subspace consisting of the homogeneous elements of degree n, than

$$EW=\bigoplus_{n\in\mathbb{Z}} EW^n$$

and

$$EW^m EW^n \subset EW^{m+n}.$$

This grading is the second important tool.

The following proposition will be used more than once in the following sections.

PROPOSITION 3. Let M be a *subspace* of EW with the property that $x_i M \subset M$ for $1 \le i \le 4$, than M is an *ideal* of EW.

PROOF. All elements of the free algebra $L(x_1, \ldots, x_4)$ can be expressed as linear combinations of words of the form $x_{i_n}(x_{i_{n-1}}(\ldots(x_{i_2}x_{i_1})\ldots)$ with $1 \le i_k \le 4$ for $1 \le k \le n$ (see ref. (5)). The same hold for EW, EW being a presentation. Set

$$x \text{ for } x_{i_n}(x_{i_{n-1}}(\ldots(x_{i_2}x_{i_1})\ldots)$$

than

$$M(x_{i_{n+1}}x)\subset(Mx_{i_{n+1}})x+x_{i_{n+1}}(Mx)\subset Mx+x_{i_{n+1}}M\subset M+M=M.$$

§ 3. THE "RADICAL" H

If L is a finite dimensional Liealgebra, R its radical and DL its derived algebra, then $R=(DL)^{\perp}$, that means that R is the orthoplement of DL with regard to the Killingform (see ref. (7)). The radical of the eight-dimensional algebra of table 1 is

$$(x_1+x_5+\lambda x_6,\ x_2-x_6,\ x_3,\ x_4-\lambda x_5-\lambda^2 x_6,\ x_8). \tag{9}$$

If we look at table 1 then we see that

$$x_5+\lambda x_6=-x_5x_7$$

so it is reasonable to replace

$$x_1+x_5+\lambda x_6 \text{ by } x_1-x_5x_7.$$

This is confirmed when we consider the radical of the eleventh dimensional algebra. We find then

$$x_1-x_{10} \text{ and that } =x_1-x_5x_7.$$

With additional help of the fourteenth-dimensional algebra we can guess that

$$H=(x_1-x_{10},\ x_2-x_6,\ x_3,\ x_4+x_{12},\ x_8)$$

is a solvable ideal of EW.

This guess is not bad, for if we replace x_9 by $\lambda(x_7-x_8)$ and use the fact that $x_{10}=x_5x_7$ and $x_{12}=x_5x_9$ we find the radical (9) again.

When we commute the elements of H with the basic letters $x_1,\dots,x_4$ then we find either 0 or x_8. Prop. 3 tells us then that *H is an ideal of EW*.

Table 2 shows that H is isomorphic with the algebra H of Shadwick (ref. (3)) of which he has shown that it is obtained as a presentation of $L(x_1,\dots,x_4)$ if one adds the relator x_7-x_8 to the relators (8). In other words: *H is isomorphic with $EW/(x_7-x_8)$* where now (x_7-x_8) is the ideal of EW generated by x_7-x_8.

The elements of EW which can be represented by linear combinations of multi-letter words form clearly an ideal I. It is evident that

$$EW=H+I.$$

Table 2 tells us that x_8 commutes with $x_1,\dots,x_4$ and that means that x_8 is a central element of EW.

All elements of (x_7-x_8) can be expressed by linear combinations of words of at least two letters, for $x_7=x_2x_1$ and $x_8=x_4x_3$.

We had already that H is isomorphic with $EW/(x_7-x_8)$ and that means that codim $(x_7-x_8)=5$.

$$x_{10}=x_5x_7,\ x_6=x_2x_7,\ x_{12}=x_5x_9=(x_1x_7)x_9.$$

That relations imply that $x_1,\dots,x_4\in H+(x_7-x_8)$ or that

$$EW=H+(x_7-x_8).$$

Table 2
Gragert's table (table 2). The element in row i and column j is $x_i x_j$.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}
x_1		$-x_7$	0	0	x_9	$-x_7+x_8$	x_5	0	$-x_{11}$	0	x_{14}	0	x_{15}	x_{13}	x_{17}	$-x_{15}$	$-x_4x_{13}$
x_2			0	$-x_9$	$-x_7+x_8$	0	x_6	0	$-x_5-x_{10}$	x_7-x_8	x_9	x_9	$-x_{14}$	$-x_{11}+x_{12}$	x_{16}	0	$-x_{15}$
x_3				$-x_8$	0	0	0	0	0	0	0	0	0	0	0	0	0
x_4					x_{14}	x_9	x_{11}	0	x_{13}	0	$-x_{15}$	0	x_4x_{13}	$-x_{17}$	$-x_{12}x_{15}$	x_4x_{16}	x_4x_{17}
x_5						x_7-x_8	x_{10}	0	x_{12}	$-x_9$	0	x_{14}	0	$-x_{13}-x_{16}$	$-x_{11}x_{14}$	x_{15}	$x_4x_{13}+x_4x_{16}$
x_6							x_6	0	$-x_5-x_{10}$	x_7-x_8	x_9	x_9	$-x_{14}$	$-x_{11}+x_{12}$	x_{16}	0	$-x_{15}$
x_7								0	0	$-x_5$	x_{12}	x_{11}	$x_{13}+x_{16}$	0	0	$-x_{16}$	$x_{11}x_{14}$
x_8									0	0	0	0	0	0	0	0	0
x_9										x_{11}	$x_{13}+x_{16}$	x_{13}	$x_{11}x_{14}$	0	$-x_{11}x_{13}-x_4x_{16}-x_4x_{13}$	$-x_{11}x_{14}+x_{17}$	x_9x_{17}
x_{10}											x_{14}	0	x_{15}	x_{13}	x_{17}	$-x_{15}$	$-x_4x_{13}$
x_{11}												$-x_{15}$	$x_{11}x_{13}$	$x_{11}x_{14}$	$x_{11}x_{15}$	$x_{11}x_{16}$	$x_{11}x_{17}$
x_{12}													$-x_4x_{13}$	x_{17}	$x_{12}x_{15}$	$-x_4x_{16}$	$-x_4x_{17}$
x_{13}														x_9x_{17}	$x_{13}x_{15}$	$x_{13}x_{16}$	$x_{13}x_{17}$
x_{14}															$x_{14}x_{15}$	$x_{14}x_{16}$	$x_{14}x_{17}$
x_{15}																$x_{15}x_{16}$	$x_{15}x_{17}$
x_{16}																	$x_{16}x_{17}$
x_{17}																	

Table 2 shows that

$$x_7 H = \{0\}.$$

We are thus led to the following theorem.

THEOREM 1. $EW = H \times (x_7 - x_8)$.

PROOF. We are left with proving that

$$H \cap (x_7 - x_8) = \{0\}.$$

Suppose there exists an element $x \neq 0$ with

$$x \in H \cap (x_7 - x_8).$$

Then codim $(x_7 - x_8) < 5$ whereas codim $(x_7 - x_8) = 5$.

§ 4. THE IDEAL $(x_7 - x_8)$

According to the theorem of Levi-Malcev every finite dimensional Liealgebra splits up in its radical and a semisimple subalgebra, the so-called *Levisubalgebra*. The latter can be $\{0\}$. If the Levisubalgebra is an ideal then the Liealgebra is the direct product of its radical and the Levisubalgebra.

The eight-dimensional algebra of table 1 has the radical (9) and one Levisubalgebra, in this case an ideal,

(11) $\quad (x_5, x_6, x_7 - x_8)$.

This ideal is an $\mathfrak{sl}2$. The *standardform* is

$$(2x_5 + \lambda x_6,\ 2(x_7 - x_8),\ x_6),$$

that means that with $x = 2x_5 + \lambda x_6$, $h = 2(x_7 - x_8)$ and $y = x_6$ the following relations hold:

(12) $\quad hx = 2x,\ hy = -2y$ and $xy = h$.

From table 1 it follows that $x_5 + \lambda x_6 = -x_5 x_7 = -x_{10}$ so that a guess of an $\mathfrak{sl}2$ is

(13) $\quad (x_5 - x_{10},\ 2(x_7 - x_8),\ x_6)$

and this is confirmed by table 2.

In § 5 it will turn out that this is the only $\mathfrak{sl}2$ in EW. We call this algebra A_1 as is usually done in the classification-theory of semisimple Liealgebras.

REMARK. $\deg(x_5 - x_{10}) = 1$, $\deg(x_7 - x_8) = 0$ and $\deg(x_6) = -1$.

The basic letters with strictly negative degree are

$$x_3 \text{ with degree } -3 \text{ and } x_2 \text{ with degree } -1.$$

Now $x_3 \in H$ and so commutes with all elements of $(x_7 - x_8)$. In all finite dimensional representations the elements $x_5 - x_{10}$ and x_6 are represented by nilpotent matrices and $2(x_7-x_8)$ by a semisimple (diagonizable) matrix (see ref. (8)). It is therefore to expected that x_2 is nilpotent because $x_2-x_6 \in H$ so that

$$(x_2-x_6)((x_7-x_8))=\{0\}.$$

Table 2 shows indeed that

$$x_2^3 x_i=0,\ 1\le i\le 4.$$

Thus the letters with negative degree are both nilpotent, i.e. $x_2^n x=0$ and $x_3^n x=0$ for all $x\in(x_7-x_8)$ with an n depending on x. This follows by induction with application of *Leibniz' rule*:

$$(14)\qquad a^n(bc)=\sum_{k=0}^{n}\binom{n}{k}(a^{n-k}b)(a^k c).$$

One is led to the conjecture that there are few elements with negative degrees. This is strengthened by table 2.

We will prove that the only elements with strictly negative degree are x_2, x_3 and x_6.

In table 2 one can find only two elements with degree 0, namely x_7 and x_8, three elements with degree 1, namely x_1, x_5 and x_{10} and one element of degree 2, namely x_9.

It seems that all other elements have a degree >0.

The natural question arises: "Is there in (x_7-x_8) an ideal of EW such that (x_7-x_8) as a linear space is split up in A_1 and that ideal?"

As $x_1=x_1-x_{10}-\frac{1}{2}(x_5-x_{10})+\frac{1}{2}(x_5+x_{10})$, $x_1(x_5+x_{10})=x_9$ and $x_2x_9=-x_5-x_{10}$ it looks that $(x_5+x_{10})=(x_9)$ is a good candidate for that ideal.

Now one can worries oneself whether there can pop up elements in (x_9) with negative degree because of x_2.

The following proposition tells us that things are not so bad.

PROPOSITION 4. The linear space (x_7-x_8) is equal to

$$A_1\oplus(x_9)$$

and (x_9) consists of linear combinations of the words which have only the "letters" x_5-x_{10} and x_5+x_{10} with exception of the word x_5-x_{10} itself.

PROOF. Let M be the subspace of (x_7-x_8), generated by the words of the form $a_{i_n}(a_{i_{n-1}}\dots(a_{i_2}a_{i_1})\dots)$ with $1\le i_k\le 2$ for $1\le k\le n$ and $a_1=x_5-x_{10}$, $a_2=x_5+x_{10}$.

We shall show that M is an ideal in five steps.

STEP 1. $x_1M\subset M$.

PROOF.

$$x_1=(x_1-x_{10})-\tfrac{1}{2}(x_5-x_{10})+\tfrac{1}{2}(x_5+x_{10})=(x_1-x_{10})-\tfrac{1}{2}a_1+\tfrac{1}{2}a_2.$$

Now $x_1-x_{10}\in H$, so $(x_1-x_{10})\,M=\{0\}$. The rest is clear.

STEP 2. $hM \subset M$ with $h = 2(x_7 - x_8)$.

PROOF. $ha_1 = 2a_1$ and $ha_2 = -2a_2$. If $ha = \lambda a$ and $hb = \mu b$ with scalars λ and μ than according to Jacobi $h(ab) = (\lambda + \mu)ab$. The result follows.

STEP 3. $x_2 M \subset M$.

PROOF. $x_2 a_1 = -h$ and $x_2 a_2 = 0$. The result follows from step 2 by induction. N.B. $a_1 \notin M$.

STEP 4. M is a subalgebra.

PROOF. $(x_7 - x_8)$ is generated by words which can be made by the letters x_1 and x_2 only, for $x_7 = x_2 x_1$ and $x_8 = x_1 x_6 + x_7$ with $x_6 = x_2 x_7$ (see (7)). The result follows from prop. 3 and steps 1 and 3.

STEP 5. $x_i M \subset M$ for $i = 3, 4$.

PROOF. $x_3 a_1 = x_3 a_2 = 0$, $x_4 a_1 = x_4 a_2 = x_{14} = \frac{1}{4} a_1 (a_2^2 a_1)$ according to table 2. Now $x_i(ab) \in M$ if $x_i a$ and $x_i b \in M$ according to Jacobi and step 4. The result follows by induction.

Prop. 3 tells us that M is an ideal.

Now $x_9 = \frac{1}{2} a_2 a_1$ and $x_2 x_9 = -a_2$, so $M = (x_9)$.

Further is $x_5 + x_{10}$ *the only element of* (x_9) *with degree* 1, *all other elements have degree* ≥ 2, so that

$$A_1 \cap (x_9) = \{0\}$$

and it is clear that $A_1 + (x_9) = (x_7 - x_8)$.

If D is the morphism $A_1 \to$ the algebra of derivations of (x_9) defined by $(Da)(x) = ax$ for $a \in A_1$ and $x \in (x_9)$ we see that $(x_7 - x_8)$ is the semidirect product

$$A_1 \times_D (x_9).$$

We have therefore proven

THEOREM 2. $EW = H \times (A_1 \times_D (x_9))$.

REMARK. The fact that $(x_7 - x_8)$ is a *nondirect* product of A_1 and (x_9) is the cause that Wahlquist and Estabrook hit the jackpot!

§ 5. THE IDEAL (x_9)

We announced already that (x_9) has no $\mathfrak{sl}2$ as a subalgebra. In fact (x_9) *contains no finite dimensional semisimple algebra* at all, except $\{0\}$ of course.

Suppose $B \neq \{0\}$ is semisimple and $B \subset (x_9)$ than there is a semisimple element $h \neq 0$; a nilpotent $x \neq 0$; both in B and a complex scalar $\mu \neq 0$ such that

$$hx = \mu x. \tag{15}$$

Let $\deg(h)=m$ and $\deg(x)=n$. We can develop h and x in their homogeneous components

$$h=h_1+\ldots+h_m \text{ and } x=x_1+\ldots+x_n$$

with $\deg(h_i)=\deg(x_i)=i$.

Let the component of h of lowest degree $\neq 0$ be h_k and x_l that of x. Then the component of lowest degree and $\neq 0$ of hx has a degree $\geq k+l$ and this is not possible in view of (15).

It is now easy to prove the theorem of Shadwick.

PROPOSITION 5 (*theorem of Shadwick*): $\dim(x_9)=+\infty$.

PROOF. Suppose $\dim(x_9)<+\infty$, then according to the theorem of Levi-Malcev (x_9) must be solvable. It follows that every homomorphic image of EW can only have as semisimple Levisubalgebra an $\mathfrak{sl}2$ or $\{0\}$, but we know that the eleventh-dimensional algebra has a sixth dimensional semisimple subalgebra and the fourteenth dimensional algebra a ninth dimensional semisimple subalgebra (see ref. (2)).

COROLLARY. Every finite dimensional subalgebra of (x_9) is solvable.

An important aspect of the grading of EW is the relation

$$EW^0EW^n\subset EW^n.$$

It follows that the EW^n are eigenspaces of the elements of EW^0. These are $x_8\in H$ and $2(x_7-x_8)\in A_1$. x_8 has only the eigenvalue 0 so that $2(x_7-x_8)$ is the only interesting element of EW^0.

The ideal (x_9) can be build with x_5-x_{10} and x_5+x_{10} as known already.

From step 2 of the proof of prop. 4 it follows that all words made of x_5-x_{10} and x_5+x_{10} are eigenvectors of $2(x_7-x_8)$. Let us denote

$$2(x_7-x_8) \text{ by } h \text{ or } h_0,\ x_5-x_{10} \text{ by } y \text{ or } y_1,\ x_6 \text{ by } z \text{ or } z_{-1} \text{ and } x_5+x_{10} \text{ by } z_1;$$

the indices reflect the degrees.

Let x be an eigenvector with $hx=\lambda x$, then

$$h(yx)=(hy)x+y(hx)=2yx+\lambda yx=(\lambda+2)yx \text{ and } h(zx)=(\lambda-2)zx.$$

It turns thus out that (x_9) *is a representation of* A_1, or equivalently: (x_9) *is a* A_1*-module.*

We know already that $x_2^n x=0$ for every $x\in(x_9)$ and suitable n. This means that $x_6^n x=z^n x=0$, because $x_2-x_6\in H$.

From table 2 it follows that

$$(x_5-x_{10})^3x_i=0,\ 1\leq i\leq 4$$

so that also $(x_5-x_{10})^n x=0$ for every $x\in(x_9)$ and suitable $n\in\mathbf{N}$ depending on x or $y^n x=0$.

This means that *all irreducible representations in* (x_9) or that *all simple* A_1-*modules are finite dimensional* (ref. (8)).

Let $V \subset (x_9)$ a simple A_1-module of dimension $m+1$, then V has a basis $\{v_0, v_1, \dots, v_m\}$ with

$$(16)\qquad \begin{cases} v_i = \dfrac{1}{i!} z^i v_0 \ (i \geq 0) \\ h v_i = (m-2i) v_i \\ z v_i = (i+1) v_{i+1} \ (i \geq 0), \\ y v_i = (m-i+1) v_{i-1} \end{cases}$$

moreover

$$v_{m+1} = v_{-1} = 0$$

from which follows that

$$z v_m = y v_0 = 0, \text{ see ref. (8).}$$

The numbers $m-2i$ are called *weights*, m the *highest weight*. The weights are

$$-m, -m+2, -m+4, \dots, m-4, m-2, m.$$

Let us now make a picture of all elements of $(x_7 - x_8)$ with degree $-1, \dots, 7$. We do this using table 2 and a Hall-basis (ref. (5)) lest we forget something. We get the following:

degree ↑ \ weight →	-2	0	2
7	$-x_{11}x_{14}+x_{17}$		$x_{11}x_{14}+x_{17}$
6		x_{15}	
5	x_{16}		$2x_{13}+x_{16}$
4		x_{14}	
3	$-x_{11}+x_{12}$		$x_{11}+x_{12}$
2		x_9	
1	x_5+x_{10}		x_5-x_{10}
0		$2(x_7-x_8)$	
-1	x_6		

If we denote the element of weight -2 by

$$z_{-1},\ z_1,\ z_3,\ z_5 \text{ and } z_7,$$

the elements of weight 0 by

$$h_0,\ h_2,\ h_4 \text{ and } h_6$$

and the elements of weight 2 by

$$y_1,\ y_3,\ y_5 \text{ and } y_7$$

then it follows from table 2 that

$$\begin{aligned} z_{2n+1} &= \text{(scalar multiple of) } z_1 h_{2n}, \\ y_{2n+1} &= \text{(scalar multiple of) } y_1 h_{2n} \text{ and} \\ h_{2n+2} &= \text{(scalar multiple of) } y_1(z_1 h_{2n}) = \\ &= \text{(scalar multiple of) } z_1(y_1 h_{2n}) \text{ for } n=0,\ 1,\ 2 \text{ and } 3. \end{aligned}$$

Let $(x_7-x_8)^n$ be defined as $(x_7-x_8)\cap EW^n$ for $n=0,1,2,\ldots$

THEOREM 3. $(x_7-x_8)^{2n}$ is generated by $h_{2n}\neq 0$ and $(x_7-x_8)^{2n+1}$ by $z_1 h_{2n}\neq 0$ and $yh_{2n}\neq 0$.

$$h_{2n+2}=y(z_1 h_{2n})=z_1(yh_{2n}).$$

$$h_2 h_{2n}=0.$$

PROOF. This is true for $n=0$. Assume the truth for $m=0,1,2,\ldots,2n+1$ then we can prove the rest in four steps.

STEP 1. $y(z_1 h_{2n})=z_1(yh_{2n})$.

PROOF.

$$z(y(z_1 h_{2n}))=(zy)(z_1 h_{2n})+y(z(z_1 h_{2n})).$$

The weight of $z_1 h_{2n}$ is -2 so that

$$(zy)(z_1 h_{2n})=2z_1 h_{2n}.$$

The weight of $z(z_1 h_{2n})$ is -4 and its degree $2n$, but h_{2n} is the only element of $(x_7-x_8)^{2n}$, so that $z(z_1 h_{2n})=0$, because weight $(h_{2n})=0$.

$$z(z_1(yh_{2n}))=z_1(z(yh_{2n})) \text{ because } zz_1=0.$$

Now h_{2n} is by induction the "middle" element of a three dimensional simple module (the lowest weight is -2). It follows from (16) that

$$z(yh_{2n})=2h_{2n}.$$

So $z(y(z_1 h_{2n})=z(z_1(yh_{2n}))$, the degree is $2n+1$ and weight is -2.

Now $z^2(y(z_1 h_{2n}))=0$ otherwise we had an element of degree $2n$ and of weight -4 again.

It follows that both $y(z_1 h_{2n})$ and $z_1(yh_{2n})$ are "middle" elements of a three dimensional simple A_1-module, so that they must be equal.

STEP 2. $y^3 z_1 = z_1^3 y = 0$.

PROOF. $x_1 x_4 = 0$ (see (8)). $x_1 - x_{10}$ and $x_4 + x_{12} \in H$.

It follows that $x_{10} x_{12} = 0$, indeed confirmed by table 2. It follows from table 2 that

$$-4x_{12} = y^2 z_1 + z_1(y z_1).$$

$y = x_5 - x_{10}$ and $z_1 = x_5 + x_{10}$ so that

$$-2x_{10} = y - z_1.$$

$$0 = (y - z_1)(y^2 z_1 + z_1(y z_1)) = y^3 z_1 - z_1(y^2 z_1) + y(z_1(y z_1)) - z_1^2(y z_1) =$$
$$= y^3 z_1 + z_1^3 y.$$

Applying of h yields:

$$0 = 4y^3 z_1 - 4z_1^3 y.$$

STEP 3. $y^2 h_{2n} = z_1^2 h_{2n} = 0$.

PROOF. $y^2 h_{2n} = 0$ because $y h_{2n}$ is a vector by the weight 2 that is maximal (see proof of step 1).

$$h_{2n} = z_1(y h_{2n-2}) = y(z_1 h_{2n-2}) \text{ (step 1).}$$

$$z_1^2 h_{2n} = z_1^3(y h_{2n-2}) = (z_1^3 y) h_{2n-2} + 3(z_1^2 y)(z_1 h_{2n-2}) +$$
$$+ 3(z_1 y)(z_1^2 h_{2n-2}) + y(z_1^3 h_{2n-2}).$$

Now by induction

$$z_1^2 h_{2n-2} = 0 \ (z_1^2 h_0 = 0) \text{ and by step 2 } z_1^3 y = 0.$$

It follows that

$$z_1^2 h_{2n} = 3(z_1^2 y)(z_1 h_{2n-2}).$$

On the other hand:

$$z_1^2 h_{2n} = z_1^2(y(z_1 h_{2n-2})) = (z_1^2 y)(z_1 h_{2n-2})$$

and the result follows.

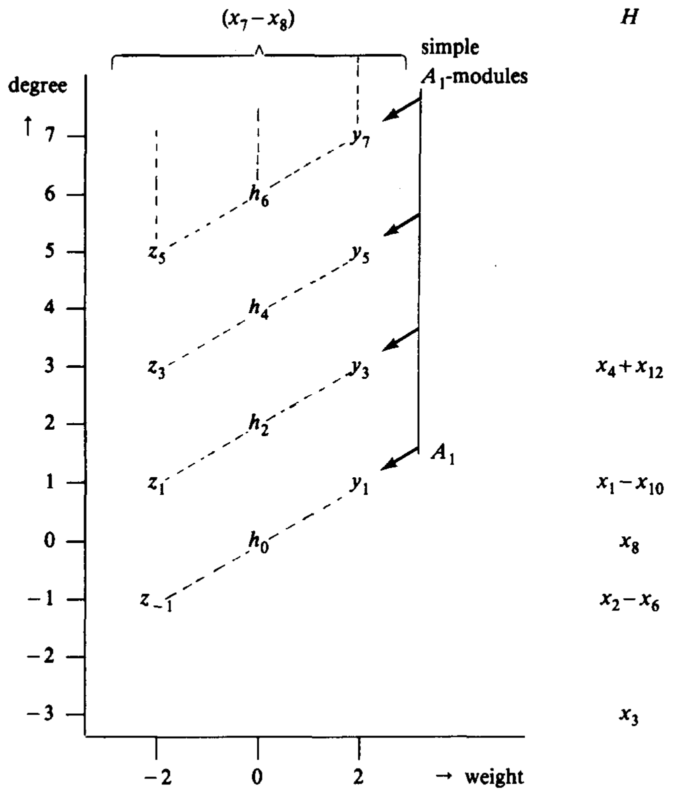
STEP 4. (x_9) is a presentation of the free Lie Algebra $L(x_5 - x_{10}, x_5 + x_{10})$, so every element of (x_9) of degree $2n+2$ must be a linear combination of the words $z_1^2 a_1$, $z_1(y a_2)$, $y(z_1 a_3)$ and $y^2 a_4$ (see proof of prop. 3). $\mathrm{Deg}(a_i) = 2n$, so a_i must be a scalar multiple of h_{2n} $(1 \leq i \leq 4)$. It follows from step 1 and step 3 that $(x_7 - x_8)^{2n+2}$ is generated by just one element. This element is denoted by h_{2n+2}. Its weight is 0. The same reasoning tells us that $(x_7 - x_8)^{2n+3}$ is generated by $z_1 h_{2n+2}$ and $y h_{2n+2}$.

The fact that h_{2n+2}, $z_1 h_{2n+2}$, and $y h_{2n+2}$ are non zero is a consequence of Shadwick' theorem (prop. 5).

Indeed, if one of them vanishes than all elements of higher degree vanish too, which implies dim $EW < \infty$.

The last statement of the theorem follows from the result of step 1.

We can now draw a picture of EW:



With help of a scaling we can prove the following formulae.

$$h_{2m} y_{2n+1} = 2y_{2m+2n+1}$$

$$h_{2m} h_{2n} = 0 \text{ for } m \text{ and } n \in \mathrm{N}.$$

$$h_{2m} z_{2n-1} = -2z_{2m+2n-1}$$

$$y_{2n+1} z_{2m-1} = h_{2m+2n}$$

$$z_{2m-1} z_{2n-1} = y_{2m+1} y_{2n+1} = 0.$$

We know already that

$$z_{-1} = z = x_6,\ h_0 = h = 2(x_7 - x_8),\ y_1 = y = x_5 - x_{10}$$

$$z_1 = x_5 + x_{10} \text{ and that } h_2 = yz_1 = -2x_9.$$

We redefine y_{2n+1} by induction to be $\frac{1}{2} h_2 y_{2n-1}$. h_{2n} is then redefined as $y_{2n+1} z$ and z_{2n-1} as $-\frac{1}{2} h_{2n} z$. Let in (16)

$$v_0 = y_{2n+1},\ v_1 = -h_{2n},\ v_2 = -z_{2n-1} \text{ and } m = 3.$$

It follows that $h_{2n} = yz_{2n-1}$ and that $y_{2n+1} = \frac{1}{2}h_{2n}y$.

$$z_{2n+1} = -\tfrac{1}{2}h_2 z_{2n-1}.$$

PROOF.

$$z_{2n+1} = -\tfrac{1}{2}h_{2n+2}z = -\tfrac{1}{2}z^2 y_{2n+3} = -\tfrac{1}{4}z^2(h_2 y_{2n+1}) =$$

$$= -\tfrac{1}{4}(z^2 h_2)y_{2n+1} - \tfrac{1}{2}(zh_2)(zy_{2n+1}) - \tfrac{1}{4}h_2(z^2 y_{2n+1}) =$$

$$= -\tfrac{1}{2}h_{2n}(zh_2) + \tfrac{1}{2}h_2 z_{2n-1} \text{ because } z^2 h_2 = 0.$$

Now $-\frac{1}{2}h_{2n}(zh_2) = -\frac{1}{2}(h_{2n}z)h_2$ for $h_2 h_{2n} = 0$ according to theorem 3.

$$-\tfrac{1}{2}(h_{2n}z)h_2 = z_{2n-1}h_2 = -h_2 z_{2n-1}.$$

$$y_{2n+1}z_{2m-1} = h_{2m+2n}.$$

PROOF. For $m=0$ this is true for all n. Let us assume the validity for $2m-3$ and all n.

$$y_{2n+1}z_{2m-1} = -\tfrac{1}{2}y_{2n+1}(h_2 z_{2m-3}) = -\tfrac{1}{2}(y_{2n+1}h_2)z_{2m-3} -$$

$$-\tfrac{1}{2}h_2(y_{2n+1}z_{2m-3}).$$

Now $y_{2n+1}z_{2m-3}$ is a scalar multiple of $h_{2n+2m-2}$ so by theorem 3

$$y_{2n+1}z_{2m-1} = -\tfrac{1}{2}(y_{2n+1}h_2)z_{2m-3} = y_{2n+3}z_{2m-3} = h_{2m+2n}.$$

In the same way one can prove that

$$h_{2n}z_{2m-1} = -2z_{2n+2m-1} \text{ and that } h_{2n}y_{2m+1} = 2y_{2n+2m+1}.$$

$$h_{2m}h_{2n} = 0.$$

PROOF. This is true for $m = 0, 1$ and all n.

$$h_{2m}h_{2n} = (yz_{2m-1})h_{2n}$$

by what we already proved, so

$$h_{2m}h_{2n} = -(h_{2n}y)z_{2m-1} - y(h_{2n}z_{2m-1}) =$$

$$= -2y_{2n+1}z_{2m-1} + 2yz_{2n+2m-1} = -2h_{2n+2m} + 2h_{2n+2m} = 0.$$

$z_{2m-1}z_{2n-1} = y_{2m+1}y_{2n+1} = 0$ for there are no elements of weight -4 or weight 4.

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